MATH 245 F16, Exam 1 Solutions

1. Carefully define the following terms: irreducible, Division Algorithm theorem, (logical) equivalence, Conditional Interpretation semantic theorem

   Let $n \in \mathbb{Z}$. We call n irreducible if it is not zero, not a unit, and not reducible. The Division Algorithm theorem states: Let $a, b \in \mathbb{Z}$ with $b \geq 1$. Then there are unique $q, r \in \mathbb{Z}$ with $a = bq + r$ and $0 \leq r < b$. Propositions p, q are (logically) equivalent if they always have the same truth value. The Conditional Interpretation theorem states that for any propositions p, q, we have $p \to q \equiv q \vee \neg p$.

2. Carefully define the following terms: converse, Disjunctive Syllogism semantic theorem, predicate, counterexample

   The converse of conditional proposition $p \to q$ is the proposition $q \to p$. The Disjunctive Syllogism semantic theorem states that, for any propositions p, q, we have $p \vee q, \neg p \vdash q$. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from some domain. A counterexample is a particular domain value for some universally quantified variable, which makes the associated predicate (and hence entire proposition) false.

3. Let $a \in \mathbb{Z}$. Suppose that a is odd. Prove that a^2 is odd.

   Since a is odd, there is some integer n with $a = 2n + 1$. We have $a^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Since $2n^2 + 2n \in \mathbb{Z}$, a^2 is odd.

4. Let $a, b, c \in \mathbb{Z}$. Suppose that $a|b$ and $b|c$. Prove that $a|c$.

   Since $a|b$, there is some $n \in \mathbb{Z}$ with $b = na$. Since $b|c$, there is some $m \in \mathbb{Z}$ with $c = mb$. Combining, $c = m(na) = (mn)a$. Since $mn \in \mathbb{Z}$, $a|c$.

5. Simplify $\neg((p \to q) \to ((\neg r) \vee p))$ as much as possible. (i.e. where only basic propositions are negated)

   Step 1: Apply conditional interpretation twice to get $\neg(\neg(q \vee \neg p) \vee ((\neg r) \vee p))$.
   Step 2: Apply De Morgan's law and double negation: $(q \vee \neg p) \wedge \neg((\neg r) \vee p)$.
   Step 3: Apply De Morgan's law and double negation: $(q \vee \neg p) \wedge (r \wedge \neg p)$.
   Optional: By addition, $\neg p \vdash q \vee \neg p$, so simply $r \wedge \neg p$.

6. Simplify $\neg(\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists z \in \mathbb{R}, x \leq z < y^2)$ as much as possible. (i.e. where nothing is negated)

   Step 1: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \forall z \in \mathbb{R}, \neg(x \leq z < y^2)$. Note that $x \leq z < y^2 \equiv (x \leq z) \wedge (z < y^2)$.
   Step 2: Apply De Morgan's Law: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \forall z \in \mathbb{R}, (x > z) \vee (z \geq y^2)$.

   Note: There is no way to write $(x > z) \vee (z \geq y^2)$ as a double inequality, you must use $\vee$ or similar.

7. Prove or disprove: $\forall x \in \mathbb{R}, \lfloor x^2 \rfloor \geq x$.

   The statement is false, and we need a counterexample for the disproof. One such is $x^\star = \frac{1}{2}$. We have $\lfloor (x^\star)^2 \rfloor = \lfloor \frac{1}{4} \rfloor = 0 < \frac{1}{2} = x^\star$.

8. Use semantic theorems to prove the modus tollens semantic theorem.

   The modus tollens theorem states: $p \to q, \neg q \vdash \neg p$. We will prove this directly; hence we take as hypotheses $p \to q, \neg q$. Using conditional interpretation on $p \to q$, we conclude $q \vee \neg p$. Using disjunctive syllogism on $q \vee \neg p$ together with $\neg q$, we get $\neg p$.

9. Use a truth table to prove that $p \leftrightarrow q \equiv (p \wedge q) \vee ((\neg p) \wedge (\neg q))$.

p	q	$p \leftrightarrow q$	$p \wedge q$	$\neg p$	$\neg q$	$(\neg p) \wedge \neg q$	$(p \wedge q) \vee ((\neg p) \wedge (\neg q))$
T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	T	T	T

   The theorem follows because the third and eighth column agree.

10. Use semantic theorems to prove that $p \leftrightarrow q \vdash (p \wedge q) \vee ((\neg p) \wedge (\neg q))$.

   We will use a direct proof. By a theorem from the text[1] we have $p \leftrightarrow q \vdash (p \to q) \wedge (q \to p)$. By conditional interpretation twice, this yields $\underbrace{(q \vee \neg p)} \wedge (p \vee \neg q)$. By distributivity, this yields $((\underbrace{q \vee \neg p}) \wedge p) \vee ((\underbrace{q \vee \neg p}) \wedge \neg q)$. By distributivity twice more, this yields $((q \wedge p) \vee ((\neg p) \wedge p)) \vee ((q \wedge \neg q) \vee ((\neg p) \wedge \neg q))$. But by another theorem from the text[2], we know that $q \wedge \neg q \equiv F \equiv (\neg p) \wedge p$. This yields $((q \wedge p) \vee F) \vee (F \vee ((\neg p) \wedge \neg q))$. By disjunctive syllogism twice, we get $(q \wedge p) \vee ((\neg p) \wedge \neg q)$. Lastly, by symmetry of $\wedge$, we get $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$.

[1] called Theorem 2.17, and also Exercise 2.14.

[2] called Theorem 2.10, and also Exercise 2.3.